DM 2013

# IV INTERNATIONAL WORKSHOP ON DIRECT METHODS

1-2 October

# BOOK OF ABSTRACTS

UNIVERSITY MEDITERRANEA
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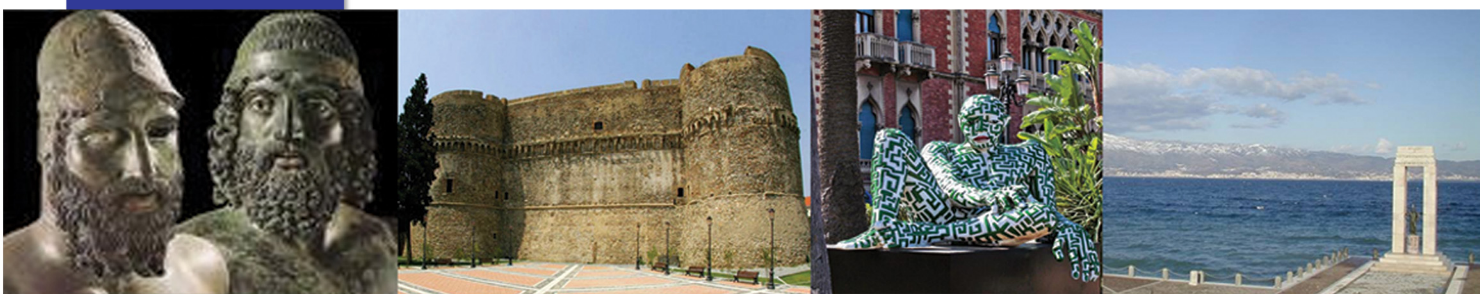

# An efficient algorithm for shakedown analysis based on equality constrained sequential quadratic programming

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*Keywords:* Shakedown, incremental analysis, mixed finite element.

In this work the algorithm proposed in [1] for incremental elastoplastic analysis is extended and applied to shakedown. Assuming rate independent associated materials, the Melan static theorem is transformed into a discrete Mathematical Programming problem by using the three field mixed finite element proposed in [3] where independent interpolations of displacement, stress and plastic multiplier are adopted. The convex optimization problem so obtained is then solved by means of a proximal point algorithm using a sequence of subproblems or steps as in [2]. The solution of each step is obtained by an Equality Constrained Sequential Quadratic Programming (EC-SQP) technique maintaining, in a manner different from the dual decomposition method used in [2], all the variables of the problem and performing a consistent linearization of all the equations. This allows the iterations to naturally evolve towards the solution. EC-SQP makes it possible to decompose the optimization problem at the finite element level by solving a series of small optimization sub-problems.

It is well known that the efficiency of the optimization methods used in shakedown analysis is highly dependent on the number of load cases that are exponentially linked with the number of constraints of the mathematical programming problem. In this work an algorithm that selects the essential constraints by an appropriate evaluation of the polytope (zonotope) defined by the elastic stresses associated to the load combinations is proposed.

The actual version of the algorithm improves that in [2] not only with respect to computational cost but also with respect to the accuracy. The finite element used, in fact, is capable of accurately evaluating the plastic multiplier value which is highly dependent on the accuracy of the elastic solution. Also in the case of ratcheting, when the plastic behaviour becomes predominant, the finite element used ensures great accuracy as shown in [3].

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# Yield design of periodically heterogeneous plates: a computational homogenization approach for the determination of macroscopic strength criteria

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This work addresses the determination of the overall strength capacities of periodically heterogeneous plates within a yield design framework. A homogenization procedure and related numerical tools are proposed to compute macroscopic strength criteria expressed in terms of generalized forces (membrane and bending solicitations). Illustrative applications focus, notably, on reinforced concrete slabs in fire conditions.

First, the general principles of homogenization in yield design are presented. As regards plates which are periodically heterogeneous in their own plane, two different small length scales arise, namely, the plate thickness and the characteristic length of the in-plane heterogeneities. The present work is devoted to the case when these two scales are of the same order. Following a modelization of the unit cell as a three-dimensional continuum, an auxiliary yield design problem with proper boundary conditions is formulated on the unit cell. The resolution of this auxiliary problem by the static or the kinematic approach leads to two equivalent definitions of the macroscopic strength criterion, which is formulated in the six-dimensional space of macroscopic bending moments and membrane forces.

A numerical resolution of the auxiliary problem by a static approach is presented, making use of simple 3D equilibrium finite elements. A particular emphasis is put on the link between the local strength criterion of steel and concrete and the resulting optimization problem, which can be formulated, either as a second-order cone programming (SOCP) problem or, more generally, as a semi-definite programming (SDP) problem.

As a first illustrative example, the case of a concrete slab reinforced by one unidirectional layer of steel bars is investigated. The corresponding numerical macroscopic strength criterion compares very well to a simple analytic formulation in the case of a small steel volumic fraction. The second illustrative example investigates the influence of fire conditions on the strength capacities of reinforced concrete slabs. The temperature gradient in the plate thickness induced by the fire deteriorates the local concrete and steel yield strengths, resulting in a decrease of the overall strength capacities of the plate which can be numerically computed. A series of experiments determining the bending strength capacities of different reinforced concrete slabs in fire conditions are confronted to the predictions of the homogenization approach.

# A statistical evaluation of the ultimate strength of WC-Co using a direct approach

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In this paper, a method is presented to determine the ultimate strength of composites with heterogeneous microstructure by combining direct methods with statistical evaluation. The method is demonstrated on a WC-Co-alloy considered as non-periodic composite. This way, the well-established lower-bound of direct methods to periodic heterogeneous media is extended to materials with random microstructure. For this class of materials, it is unlike to construct a representative volume element (RVE) to study the global material behavior: due to the difference of morphology among samples, despite having identical size and constituents, the scatter of their associated mechanical behavior is still remarkable. To overcome this difficulty, an algorithm is developed which automatically converts scanning electron microscopy (SEM) images to 2D RVE model. By adopting this algorithm, limit analysis is performed on a group of RVE models built from the real microstructure and their results are interpreted statistically. Besides, in order to validate obtained results, conventional incremental analyses are performed on selected samples for comparison.

Since it is widely acknowledged that for particulate reinforced composite such as WC-Co, both plane stress and plane strain idealization will cause inaccuracy in the prediction, limit analysis is performed as well on a thin plate 3D model, namely 2.5D model. The difference between plane stress, plane strain, and 2.5D model in the sense of bearing an inelastic loading is exploited and discussed in detail.

# A stress variational macroscopic model for von Mises poroplastic materials

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The main objective of this work is to formulate a very new statically-based model of ductile porous materials having a von Mises matrix. In contrast to the Gurson's well-known kinematical approach applied to a hollow sphere, the proposed study proceeds by means of a statical limit analysis procedure. Its development and implementation require the choice of an appropriate trial stress field. The starting point is Hill's variational principle for rigid plastic matrix. The use of a Lagrangian multiplier allows satisfying the plastic criterion in an average sense. Due to the relaxation of the licit character of the trial stress fields, the criterion could be seen only as a quasi-lower bound.

The proposed trial stress field, complying with internal equilibrium, is composed of a heterogeneous part (exact solution for the stress field in the hollow sphere under pure hydrostatic loading) to which is added a uniform deviatoric stress field. Owing to this choice, the stress vector conditions on the void boundary are relaxed. By solving the resulting saddle point problem, we derive closed form formula which depends not only on the first and second invariant of the macroscopic stress tensor but also on the sign of the third invariant of the stress deviator. This leads to specific asymmetries of the macroscopic criterion.

The obtained results are fully discussed and compared to existing models, available numerical data and to Finite Elements results obtained from cell calculation carried out during the present study. We also provide for the new model the macroscopic flow rule as well as the porosity evolution equations which also show very original features. Finally, it is convenient to indicate that, besides the original statically-based methodology provided in the present study, an important perspective lies in the possibility now to investigate the case of porous media with a non associated matrix. Clearly enough this can be addressed by means of the Bipotential approach introduced by De Saxcé and which has already led to a generalization of classical limit analysis theorems to the context of non associated materials. In this perspective, the interest of the present study lies in that one will need for the implementation of the bipotential theory (for porous media) both the trial velocity and the trial stress fields.

# Limit state of heterogeneous materials under thermo-mechanical loading

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*Keywords:* Direct methods, heterogeneous materials, temperature dependent yield strength, temperature dependent Young's modulus

Direct methods, namely shakedown and limit analysis, are presented in this paper for the prediction of loading bearing capacity and material properties of heterogeneous materials, which are subjected to variable thermo-mechanical loads. Direct methods for heterogeneous materials concerns two scales: on the microscopic scale, the lower-bound direct methods is applied to representative volume element (RVE); on the macroscopic scale, by means of homogenization theory, the global effective material properties, as well as the loading bearing capacity are obtained by transformation from the local stress field to global admissible stress field. Moreover, the application of lower-bound direct methods is extended to composites with temperature linearly dependent yield strength and Young's modulus. Some numerical examples validate the methodology and confirm the applicability on the structure design.

# A direct method formulation for topology plastic design of continua

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Research in topology optimization of continuum structures has witnessed a considerable development during the last decades. The numerous successful applications of topology optimization in industry and the emergence of powerful dedicated topology optimization software reflect the degree of maturity this discipline has reached.

For various reasons, most of the work on continuum topology optimization has been restricted to elastic material behavior. The mathematical approaches underlying the topology design methodologies for continuum structures often rely strongly on the linear elasticity assumption. The linearity assumption is a prerequisite to the equivalence of potential energy and compliance, which is the key to a large class of methods [1]. It is also suitable for formulating interpolation schemes [2], such as SIMP, and eliminates the difficulty associated with modeling the nonlinear behavior of materials at intermediate material densities. Regardless of the degree of complexity of the material behaviour, topology optimization of continuum structures raised difficulties that took tremendous amount of work to overcome. These include the multiplicity of local minima, the issues of ill-posedness of the mathematical problem, the lack of convergence with respect to the size of the finite element mesh [3] and the difficulty associated with local stress constraints [4].

Interest in topology design of media characterized by elastoplastic or plastic behaviours remains limited, mainly because of the high computational demand of response calculation using nonlinear, iterative analyses and the increased complexity of the mathematical problem.

Unlike response oriented analyses, direct methods of limit analysis are known to require relatively lower computational effort to determine limit states in terms of either stress field or displacement/velocity field solutions. When information on the history of evolution is not needed and, say, only the limit stress field is of interest, direct methods turn out to be an adequate alternative for plastic collaspse analysis [5]. In an automated design context, where computational efficiency is a primary factor, direct methods of limit analysis present a paramount advantage. In

[6] plastic sizing design of trusses and plates was performed using integrated limit analysis and design.

The present work is precisely concerned with the integration of a direct method of limit analysis into a methodology for plastic topology design of continuum structures. The proposed formulation belongs to the microstructure (or material) approach [7]. In a plane strain framework, the initial design domain is modeled entirely with a regular finite element mesh. The design variables consist of a material density parameter attributed to each finite element. For solutions with "porous topology", i.e. with a continuous range of densities between 0 and 1, a judicious formulation of the topology design problem for minimum weight is proposed such that the design problem takes on a form similar to that of a direct static limit analysis problem. In this formulation the design constraints are restricted to a limitation on the collapse load.

The key idea of the formulation is to assume the strength or (cohesion) of the material to be proportional to the material density. The computational demand for the topology design problem using the proposed approach is consequently in the order of that of the execution of a single limit analysis. Furthermore, the topology optimization problem possesses the same mathematical properties as the direct limit analysis problem, notably, convexity and, for a Tresca material, conicity. These properties, which do not hold even for linear elasticity, have significant implications with regard to global optimality, well posedness and convergence.

The proposed topology design method is formulated using a Tresca material. It is implemented using the conic programming code MOSEK [8] for solving the mathematical programming problems expressing the plastic topology design problems. The method is illustrated and its performance is demonstrated through example problems taken from the literature.

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# Limit Analysis using a variable target value method

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The finite element formulation of the direct problem of limit analysis gives rise to a nonlinear constrained optimization problem. The plasticity criterion being convex in general, the mathematical programming problem expressing the limit analysis problem can be made convex owing to a judicious construction of the finite elements [1], [2]. For Tresca and von Mises materials the limit analysis problem can conveniently be written as a conic programming problem which can be solved efficiently using existing conic programming codes (e.g. MOSEK [3]) even for large dimensions. Medium and large size convex problems with more general forms that are not convertible to a conic from continue to pose a computational challenge.

Sequential quadratic programming methods and their decomposition based variants are known to provide better performance among existing methods in solving general nonlinear programming problems. However, their performance deteriorates in the presence of degeneracy [4].

In the present work, a method is proposed for solving large convex limit analysis problems that is aimed to be robust with respect to degeneracy. It consists of an extension of the variable target value method (VTVM) [4] developed for solving the dual of large linear programming problems exhibiting degenerate solutions. In presence of degeneracy, the VTVM has been demonstrated to ensure convergence when the search directions are chosen according to a conjugate subgradient direction strategy. Subsequently the VTVM was modified by improving the target solution and adopting a Polyak-Kelley cutting plane technique to determine the search direction [5]. In the present work, both the original and modified versions of the VTVM are adapted to solve the convex, nondifferentiable dual of the numerical static limit analysis problem. The dual function takes the form of a sum of norms and the nondifferentiability points are those where at least one of the radicands vanishes.

In order to speed up the process in its early stages, the search is started using a different method to find a good starting point for the VTVM. Two methods were proposed in [6] for this purpose. The first was based on a perturbation strategy that moves the current solution away from the nondifferentiability locus and the second on a smoothing technique. In [6] the smoothing was achieved via a penalty function. In the present work a different smoothing method is utilized. It consists in adding a positive constant to each of the radicands appearing in the objective function. The smoothened problem can then be solved using popular gradient methods that can efficiently handle large size problems.

The proposed method is tested on example limit analysis problems, including the vertical cut and the compressed bar static problems, and performance is compared with that of the commercial conic programming code MOSEK [3].

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# A new starting point strategy for shakedown analysis

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Lower bound shakedown analysis is currently implemented by the coupling of finite element methods with techniques of computational optimization. Engineering structures problems contain a large number of variables and constraints, leading to large-scale non-linear programming problems. Non-linear programming algorithms use iterative techniques to solve the problem; therefore the selection of a starting point is of crucial importance for their performance. To this goal the elastic limit solution could be applied, which yields a feasible point, since the zero residual stress identically satisfies the null space conditions.

The present study proposes a simple technique to obtain an initial feasible point with nonzero residual stresses starting from the plastic shakedown analysis (PSA). The residual stresses obtained by PSA, and which are generally infeasible, are projected into the null space of the equilibrium conditions to yield the desired feasible set of the nonzero residual stresses. The initial feasible point is weighted by a safety factor, obtained from a one-dimensional optimization problem of elastic limit type.

The applicability and appropriateness of this approach is studied by numerical comparisons.

# Limit analysis, axisymmetry and porous Coulomb problems

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*Keywords:* Limit analysis, axisymmetry, mixed method, porous Coulomb, spheroidal Gurson model.

The first purpose of this work was to develop efficient upper bound limit analysis tools for axisymmetric limit analysis (LA) problems, keeping in mind that singularity problems can occur when using classical LA methods. The second purpose concerns the prediction of the macroscopic criterion of a porous material on the basis of the hollow spheroid model with a Coulomb matrix, without any smoothing hypothesis of the plasticity criterion.

The starting point of this study is the conic expression of the Coulomb criterion used in a linearized form in [1]. Up to our knowledge, it is the first time that the problem of predicting the plasticity criterion of a porous Coulomb material with spheroidal cavities is investigated. A similar investigation was conducted for this kind of materials with von Mises matrices in the full 3D case, resulting in a formulation not easily transposable to Coulomb matrices due to the general expression of this criterion.

To obtain the lower bound results, we have adapted the iterative quadratic static approach defined in [1], with a linear programming formulation leading to a sufficiently robust code. Unfortunately this was not the case when adapting the classic kinematic approach also presented in this reference. Then, we have developed an original mixed (but rigorously kinematic) approach of the present axisymmetric Coulomb problem. The final problem ends up in a second order conic formulation solved with the efficient code Mosek, and the results are checked as fully admissible by specific post-analysis of the optimal solution fields. The resulting codes are first tested in the spherical cavity case with comparison to the corresponding results of [1] and to the exact solutions of [3]. Finally these codes are used to investigate the unexplored problem of the porous Coulomb material with oblate cavities.

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# Limit analysis applied to reinforced concrete structural elements

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*Keywords:* Limit analysis, Peak load, Upper and lower bounds, FE based procedures, Reinforced concrete structural elements, Large-scale prototypes.

This paper deals with the setup of a methodology able to detect the peak load, the failure mode and the critical (weaker) zones, if any, of reinforced concrete structural elements. Reinforcement made of either classical steel bars or fiber reinforced polymer (FRP) bars are considered. In particular, two numerical procedures, arising from the kinematic and the static approach of limit analysis theory and known as Linear Matching Method (LMM) and Elastic Compensation Method (ECM), are employed to search for *upper* and *lower bounds* to the actual peak load of the analyzed elements.

The LMM and ECM have been reformulated by the authors, within the context of concrete elements, in some recent contributions [1-2] where theoretical aspects as well as numerical results are given in detail. The main novelty of such approach is the application of the two methods to a 3D-plasticity model for concrete proposed by Menétrey and Willam [3]. Its effectiveness is indeed assured by the ductility injected by the presence of reinforcements. The *combined use* of LMM and ECM allows to bracket the real peak load values and is motivated by the assumption of a non associate flow rule for concrete, the latter due to its dilatancy.

The assumed yield surface is characterized by curved meridians in the Rendulic section and J3 dependence. A cap in compression is also considered. Three independent stress invariants, namely the Haigh-Westergaard coordinates, are employed for the treatment in principal stress space where the surface results convex and smooth. A perfect bonding between re-bars and concrete is assumed.

In the present work steel and FRP reinforced concrete *large-scale prototypes* of structural *beams*, *walls* and *slabs* are numerically analyzed to predict their peak load as well as their collapse mechanism. The obtained numerical results, compared with experimental laboratory findings available in the relevant literature, are critically discussed outlining possible future developments, strengths and weaknesses of the proposed methodology.

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# Limit states for complex constitutive relationships

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The advancement of Direct Methods and the derivation of complex constitutive relationships are subjects which have had their own separate lines of development. Direct Methods have generally concerned themselves with simple constitutive relationship (although not exclusively), and in particular, the elastic-perfectly plastic solid.

There are legitimate reasons for asking the following question. For particular classes of solids, described by constitutive relationships involving state variables, how can meaningful limit states be defined and under what conditions do such limit states exist? How do such limit states relate to the predictions of an elastic-perfectly plastic solid?

The talk discusses two classes of solids, ductile metals and geotechnical materials. For metals, beginning with state variable constitutive relationships with a formal structure consistent with thermodynamics, it is possible to understand the relationship between behaviour predicted by such relationships and simple Direct Methods. This may be achieved for both time independent plasticity and high temperature creep.

However, for geotechnical materials there are problems. In particular, a formal definition of a limit state does not always coincide with traditional strength calculations. There are issues concerning the nature of deformation fields at the limit state which seem to have been ignored.

The talk attempts to explore the possibility of a theory that joins together Direct Methods and complex constitutive relationships by defining critical issues.

# The influence of kinematical hardening on incremental collapse, alternating plasticity, and the corresponding residual stress fields

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*Keywords:* Incremental Collapse, Alternating Plasticity, Direct Methods, Limited Kinematical Hardening, Nonlinear Programming

In this paper, engineering structures are considered which are subjected to varying thermo-mechanical loading beyond the elastic limit. For these, the shakedown factors are determined, which are defined as the maximum loading factors such that the structure under consideration does neither fail due to spontaneous or incremental collapse nor due to alternating plasticity. This is done by means of the statical direct approach of *Melan* [1], who formulated a shakedown theorem for elastic-perfectly plastic and unlimited kinematical hardening continua. For many engineering problems, accounting for kinematical hardening is inevitable in order to obtain realistic results. Even more, the consideration of limited kinematical hardening is necessary in order to cover both incremental collapse (IC) and alternating plasticity (AP). In this work, the two-surface model proposed by *Weichert* and *Groß-Weege* [2] will be used to incorporate limited kinematical hardening into the shakedown analysis. The kinematical hardening is considered as a translation of the yield surface in stress space, which is described by the six-dimensional vector of back-stresses representing the motion of the yield surface's center. This motion is bounded through the introduction of a second surface corresponding to the ultimate stress.

In general, using the statical shakedown theorem leads to nonlinear convex optimization problems, which are typically characterized by large numbers of unknowns and constraints. This holds even more, when limited kinematical hardening is taken into account, because then the back-stresses need to be introduced as additional variables. In this work, these optimization problems will be solved by the interior-point algorithm IPSA which has been developed recently in [3]. The extension of this algorithm for limited kinematic hardening has been presented in [4, 5].

The two different phenomena IC and AP can also be separated in the mathematical formulation of the corresponding optimization problems, as shown e.g. in [5, 6]. While the kinematical hardening has no influence in case of AP, it has a significant impact on the shakedown load for IC. In this paper, structures will be investigated which are subjected to two different loading cases, the one leading to AP and the other one to IC. In particular, the influence of the limited kinematical hardening on the transition region between AP and IC as well as the corresponding residual stresses are investigated.

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# Extension of the RSDM to the shakedown analysis of structures

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Civil and mechanical engineering structures are generally subjected to high levels of loading. Typical examples of such structures are buildings and bridges under seismic loading on the one hand and nuclear reactors and aircraft gas propulsion engines on the other. Under all these kinds of loading, these structures are forced to develop plastic strains.

The question whether a structure can carry the applied loads is answered, mostly, on the basis of cumbersome time stepping calculations. To accomplish this task, one has to know the exact time history. A better alternative, that requires less computing time, is offered by the direct methods that may predict whether, under the given loading, the structure will become unserviceable due to collapse or excessive inelastic deformations. Moreover, if the complete time history of loading in not known, but only variation intervals of the loading is known, direct methods is the only way to establish safety margins.

Recently, a new direct method, which has been called Residual Stress Decomposition Method (RSDM), was proposed [1]. This method was used to determine the long-term effects of an elastoplastic structure subjected to a given cyclic loading time history. Any cyclic state, like elastic shakedown or alternating plasticity or ratcheting may be predicted.

The method takes into advantage the expected cyclic nature of the residual stresses at the cyclic state. It is the distribution of the residual stresses that is sought at the cyclic stress state. Thus, the unknown residual stresses are decomposed into Fourier series of cosine and sine terms multiplied by the unknown coefficients. The method constitutes an iterative procedure to find these coefficients. It is proved that an update of these coefficients may be found with the aid of the integral of the cycle time derivatives of these residual stresses inside a loading cycle. These derivatives may be estimated at discrete cycle points by enforcing equilibrium and compatibility at these points.

In the present work a novel method (RSDM-S), is presented which may be used to produce safety margins. The RSDM [1] forms the basis of this method which constitutes a new upper bound approach to provide the elastic shakedown factor for any loading domain. The main concept of the RSDM-S procedure is based on reducing iteratively the load factor by subtracting the sum of the norms of the Fourier coefficients of the sine and cosine terms of the Fourier series of the current residual stresses. The procedure stops if only constant terms appear in the Fourier series. The procedure is initialized using a load factor well above the shakedown load. This can be guaranteed if all the elements of the structure become plastic. Then, a residual stress distribution is calculated using the RSDM for this initial load factor.

The proposed iterative procedure in a typical iteration may be briefly described as:

1. Calculate the sum of the norms of the Fourier coefficients of the sine and cosine terms of the Fourier series of the current residual stresses.
2. Update the current load factor by subtracting an expression which contains this sum of norms.
3. Should this factor not differ, within some tolerance, from the previous iteration stop; otherwise
4. Find an update of the current total stress as the sum of the current residual stress and the current elastic stress that corresponds to the current load factor
5. Determine an update of the Fourier coefficients in the same way that it is done with the RSDM and update the residual stresses
6. If the current residual stresses do not differ, within some tolerance, from the previous iteration, go to step 1; otherwise go to step 4.

A perfectly plastic material with a von Mises yield surface is currently assumed. The stiffness matrix needs to be decomposed only once. The whole approach is shown to be stable and computationally efficient, with uniform convergence.

Another important issue is that the method may be used within any finite element program.

Examples of application for various structures will be presented during the workshop.

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# A Finite Element Formulation for Direct Calculation of Steady Plastic Cycle

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*Keywords:* direct steady cycle computation, finite elements, plastic ratcheting.

Whereas a structure usually shakes down to purely elastic deformation over a small number of cycles, stabilization to steady cycle in excess of shakedown can proceed quite slowly and take many tens of cycles [1]. In such case the whole stress-strain history can be cumbersome to compute using step-by-step approach. Although effective direct procedures for elastic shakedown have already been created and implemented, there are only few approaches being under development for steady plastic response, such as [2], which iteratively adjusts residual stresses. On the other hand, the problem of determining steady plastic response under prescribed cyclic loading can also be formulated as a constrained optimization problem [3].

In the system of expressions defining steady cycle, the associate flow rule can be substituted by the condition of minimization of a functional [3]. This leads to a convex equality and inequality constrained minimization problem. The Bree problem of pressurized thin-walled tube under repeated thermal loading [4] is solved in the study as a test example. The structure was discretized using rectangular quadratic finite elements. The conditions of plastic incompressibility, cycle closure and initial residual stress self-balance are enforced by means of quadratic penalties. An unconstrained mathematical optimization problem formulated using logarithmic barrier functions for the inequality conditions of admissibility of the total stress was solved by means of Newton's method together with exact line search.

It has been shown that the general problem of steady cycle can be solved directly by stating it as a convex mathematical optimization problem with the use of finite element discretization to capture all the parameters of a steady cycle, even though the simple optimization approach used in the study is for demonstration only and is not claimed to be efficient for the real problem.

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# Multimode failure and probabilistic shakedown analysis of shells

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The plastic collapse limit and the shakedown limit which define the load-carrying capacity of structures are important in assessing the structural integrity. Due to the high expenses of experimental setups and the time consuming full elastic-plastic cyclic loading analysis, the determination of these limits by means of numerically direct plasticity methods has been of great interest to many designers. Moreover, a certain evaluation of structural performance can be conducted only if the uncertainty of the actual load-carrying capacity of the structure is taken into consideration since all resistance and loading variables are random in nature. To ensure the safety of the structures to be designed, two approaches are normally used. (1) The classical approach fixes the values of the safety factors and chooses the values of the design variables to satisfy the safety conditions. All the variables involved are then assumed to be deterministic and fixed to particular quantiles, i.e. mean value or characteristic values. (2) The probability-based approach deals directly with realistic random variables to find the global probability of failure as the basic design criterion. Obviously, the later problem is more difficult since the evaluation of the probability of failure is not an easy task, especially when the structure has more than one failure mode (multimode failure or multiple design points). In this case, analysis of the structural system is required to evaluate the safety of the structure as a whole [1]. To handle problems of this kind, the real structure is sometimes modelled by an equivalent system in such a way that all relevant failure modes can be treated [2].

The reliability analysis of plates and shells with respect to plastic collapse or to inadaptation was formulated on the basis of limit and shakedown theorems [3]. The technique was based upon an upper bound approach using the re-parameterized exact Ilyushin yield surface and nonlinear optimization procedures. Based on a direct definition of the limit state function, the non-linear problems may be efficiently solved by using the First and Second Order Reliability Methods (FORM/SORM). In order to get the design point, a non-linear optimization was implemented.

The non-linear optimization algorithm which was developed in [3] is guaranteed to converge to a minimum-distance point on the limit state surface, provided that the limit state function is continuous and differentiable. However, as with any non-convex optimisation problem, it is not guaranteed that the solution point will be the global minimum-distance point when the system has more than one failure mode. This paper aims at extending the method developed in [3] for the probabilistic shakedown analysis of multimode-failure shell structures. A method to successively find the multiple design points of a component reliability problem, when they exist on the limit state surface, is presented. Each design point corresponds with an individual failure mode or mechanism. FORM and SORM approximations are applied at each design point followed by a series system reliability analysis to lead to improved estimates of the system failure probability.

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# Three-dimensional shakedown solutions for cross-anisotropic cohesive-frictional half-space under moving loads

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Shakedown analysis of cohesive-frictional materials under moving surface loads has been studied for nearly three decades in consideration of its application in the design of pavement. Most of the materials have been assumed to be isotropic in behaviour. However, evidence of cross-anisotropic behaviour of soil and pavement materials is mounting. Therefore, three-dimensional shakedown solutions are developed to allow for the variation of elastic and plastic material properties with direction.

A homogeneous soil half space that is cross-anisotropic with a vertical axis of symmetry is considered. In the elastic range, the behaviour of the cross-anisotropic material can be described by five elastic parameters. In consideration of plastic anisotropy, a generalised, anisotropic Mohr-Coulomb yield criterion is assumed which accounts for the directional variation of material cohesion. Following Yu [3] and Yu and Wang [2], Melan's shakedown theorem is used to derived the lower-bound shakedown solutions. Melan's shakedown theorem states that an elastic-perfectly plastic structure will shakedown if the combination of load induced elastic stress field and self-equilibrated residual stress field does not violate the yield criterion. Shakedown condition for the present problem then can be derived as: (Wang and Yu, [1])

$$f = (\sigma_{xx}^r + M)^2 + N + P \leq 0$$

where $\sigma_{xx}^r$ is self-equilibrated residual stress field; $M$, $N$ and $P$ are functions of elastic stress field, material plastic properties and dimensionless load parameter $\lambda$. By using elastic stress solutions for a cross-anisotropic half space and the self-equilibrium condition, the maximum load parameter that makes $f \leq 0$ at all points in the half-space is the lower-bound shakedown limit $\lambda_{sd}$ for the present anisotropic problem.

It is found that shakedown solutions for the present anisotropic problem are dominated by Young's modulus ratio $E_v/E_h$ for the cases of subsurface failure and by shear modulus ratio $G_{vh}/G_h$ for the cases of surface failure. Plastic anisotropy is mainly controlled by material cohesion ratio $c_v/c_h$, the rise of which increases the shakedown limit until a maximum value is reached. The anisotropic shakedown limit varies with frictional coefficient $\mu$ and the peak value may not occur at $\mu = 0$ (i.e. normal load only).

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